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ON THE INSTABILITY OF PLANE-PARALLEL FLOWS OF PERFECT FLUID

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It is shown that by Liapunov's criterion stationary flows of perfect fluid with a convex velocity profile in a channel are unstable: perturbations can increase linearly with time. It is assumed in the proofs that the perturbation smallness for $t = 0$ and $t > 0$ has the same meaning (one and the same norm is used). Depending on the choice of norm the instability is of a two-dimensional or substantially three-dimensional kind.

1. Statement of problem. In the theory of hydrodynamic instability the main interest is concentrated on strong (exponential) instability. If it is absent, it is interesting to clarify whether there is stability in the sense of some exact definition. The fundamental question is how to define the "smallness" of perturbations, and what norm to use in the definition of stability. It seems reasonable to consider perturbation $\mathbf{u}(\mathbf{x})$ small, if $\|\mathbf{u}(\mathbf{x})\|_1 = \sup|\mathbf{u}(\mathbf{x})|$ or $\|\mathbf{u}\|_2 = (\int|\mathbf{u}(\mathbf{x})|^2 d\mathbf{x})^{1/2}$ are small. Sometimes the norm $\|\mathbf{u}\|_3 = \|\mathbf{u}\|_2 + \|\operatorname{grad}\mathbf{u}\|_2$ is suitable. It is shown below, that in any of these norms all stationary plane-parallel flows of perfect fluid which have a convex velocity profile $U_0(y)$ are unstable. It is traditionally assumed that instability in linear approximation guarantees a true instability, and perturbations of specified periodicity with respect to x and z are considered (analysis of perturbations of the general kind yields similar results).

Definitions. A stationary flow $\mathbf{u}_0(\mathbf{x})$ is unstable in the meaning of norm $\|\mathbf{u}\|$, if for any n there exists a solution of the linearized equations $\mathbf{u}_n(\mathbf{x}, t)$ such that

$$\sup_{0 \leqslant t < \infty} \|\mathbf{u}_n(\mathbf{x}, t)\| \geqslant n \|\mathbf{u}_n(\mathbf{x}, 0)\| \tag{1.1}$$

The related flow $\mathbf{u}_0(\mathbf{x})$ is stable (in linear approximation), if there exists a constant K such that

$$\|\mathbf{u}(\mathbf{x}, t)\| \leqslant K \|\mathbf{u}(\mathbf{x}, 0)\| \tag{1.2}$$

Instead of (1.2) the boundedness of each individual solution is often required: for any $\mathbf{u}(\mathbf{x}, 0) = \boldsymbol{\varphi}(\mathbf{x})$

$$\|\mathbf{u}(\mathbf{x}, t)\| \leqslant C \tag{1.3}$$

This requirement is equivalent to (1.2), if $\boldsymbol{\varphi}(\mathbf{x})$ passes through some complete normalized space, and the boundedness in (1.3) is understood in the meaning of the norm of that same space (see, e.g. p.140 in [1]). If one of these conditions is not satisfied, (1.3) does not imply stability in the meaning of (1.2).

2. Input equations. Instability for $k = 0$. Setting $\psi(x, y, z; t) = \psi(y, t) e^{i(kx+mz)}$, we write the linearized equations

$$u_t + ikU_0 u + U_0' v + ikp = 0 \tag{2.1}$$

$$v_t + ikU_0 v + p' = 0, \qquad v(0, t) = v(1, t) = 0$$

$$w_t + ikU_0w + imp = 0, \qquad iku + v' + imw = 0$$

$$\left(\psi' = \frac{\partial\psi}{\partial y}, \quad \psi_t = \frac{\partial\psi}{\partial t}, \quad \mathbf{U}_0 = (U_0(y), 0, 0)\right)$$

where $\mathbf{u} = (u, v, w)$, p are small perturbations, and $\mathbf{U}_0$ is the unperturbed velocity. From (2.1) follows

$$p'' - (k^2 + m^2)\, p = -\, 2ikU_0'v, \qquad p'(0, t) = p'(1, t) = 0 \qquad (2.2)$$

$$v_t + ikU_0(y)\, v - 2ik\int_0^1 \frac{\partial G_n}{\partial y}(y, \eta)\, U_0'(\eta)\, v(\eta, t)\, d\eta = 0 \qquad (2.3)$$

System (2.1) has the following particular solutions:

1) for $k = 0$ and $m \neq 0$

$$u = f(y) - tU_0'(y)v(y), \; v = v(y), \; w = w(y), \; p = 0 \qquad (2.4)$$

$$(v' + imw = 0)$$

2) for $k \neq 0$ and $m \neq 0$

$$p = v = 0, \qquad u(y, t) = u(y, 0)\, e^{-ikU_0(y)t} \qquad (2.5)$$

$$w(y, t) = w(y, 0)\, e^{-ikU_0(y)t}$$

We introduce the notation

$$\|\mathbf{u}\|_1 = \sup_y |\mathbf{u}(y)|, \; \|\mathbf{u}\|_2 = \left(\int_0^1 |\mathbf{u}(y)|^2 dy\right)^{1/2}, \quad \|\mathbf{u}\|_3 = \|\mathbf{u}\|_2 + \|\mathbf{u}'\|_2 \qquad (2.6)$$

Then from (2.4) and (2.5) follows instability (2.1):

1) for $k = 0$ and $m \neq 0$ in the meaning of any norm

2) for $k \neq 0$ and $m \neq 0$ in the meaning of $\| u \|_3$.

Note that in the subspace $iku + v' + imw = 0$, $v(0) = v(1) = 0$ norm $\|\mathbf{u}\|_2$ is equivalent to the following:

$$\|\mathbf{u}\| = \left(\int_0^1 [|u|^2 + |v'|^2 + |w|^2]\, dy\right)^{1/2} \qquad (2.7)$$

Let us consider Eq. (2.3) and the norm

$$\|v\|_3 = \left(\int_0^1 |v'|^2 dy\right)^{1/2}, \quad v' = \frac{\partial v}{\partial y} \; (v(0) = v(1) = 0) \qquad (2.8)$$

It can be shown that instability (2.3) in this norm implies the instability of system (2.1) in norm (2.7) or $\|\mathbf{u}\|_2$.

Equation (2.3) is invariant with respect to the substitution $v(y, t) \to \bar{v}(y, -t)$. Hence to prove instability it is sufficient to indicate at least one solution which tends to zero. In fact, if $\| v(y, t_n) \| \leqslant 1/n \; \| v(y, 0) \|$, then for $v_n(y, t) = \bar{v}(y, t_n - t)$ we have (see (1.1))

$$\sup_t \| v_n(y, t) \| \geqslant n \| v_n(y, 0) \| \qquad (2.9)$$

3. Instability for $k \neq 0$ and any m. Let us assume now that the Rayleigh condition $U_0''(y) \neq 0$ which guarantees the absence of any strong instability is satisfied.

For $U_0''(y) \neq 0$ all reasonably smooth solutions of Eq. (2.3) tend to zero. This important aspect was indicated in [2, 3], however without a rigorous proof. We present such proof in abstract terms for the sake of brevity.

Let $\gamma = v'' - (k^2 + m^2)\, v$. Then from (2.3)

$$\partial\gamma/\partial t = iA\gamma, \quad A\gamma = -kU_0(y)\gamma + kU_0''(y)\, v \qquad (3.1)$$

For $U_0'' \neq 0$ operator A is self-conjugate in the metric

$$(\gamma^{(1)}, \gamma^{(2)}) = \int_0^1 \gamma^{(1)}\bar{\gamma}^{(2)} | U_0''(y)|^{-1} dy$$

and (for $k \neq 0$) has a continuous spectrum occupying the segment. As shown in [4] that spectrum is absolutely continuous. It follows from this that all solutions (3.1) $\gamma(t) = e^{iAt}\gamma(0)$ for $t \to \infty$ tend weakly to zero (i.e. $(\gamma(t), \varphi) \to 0$ for any function φ). Furthermore $v' = I\gamma$, where I is an integral operator with a piecewise continuous kernel. It is entirely continuous in $L_2(0, 1)$ consequently, $(v', v') \to 0$ and

$$\| v \|_3 = \left(\int_0^1 | v'(y,t)|^2 dy\right)^{1/2} \to 0 \quad \text{for} \quad t \to \infty$$

Thus, in the meaning of norm $\| \mathbf{u} \|_2$, system (2.1) is unstable. The proof of instability in the meaning of $\| \mathbf{u} \|_1$ is similar.

With a suitable choice $\gamma(y, 0)$ $\| v(y, t)\|_3 \leqslant C / t \cdot$ (see [2] and Note 3 below), hence it is possible to assume an approximately linear increase of perturbations with time (see (2.9)).

4. The formulation of the stability problem presented in Sect. 1 is, of course, not the only one possible. Retaining the concept of flow perturbation only for $t = 0$, it is possible to narrow the class of initial perturbations by, for instance, considering perturbations to be small, when velocities and their derivatives are small. In that case instead of (1.2) we have a condition of the kind $\| \mathbf{u}(\mathbf{x}, t) \|_2 \leqslant M \| \mathbf{u}(\mathbf{x}, 0) \|_3$, where M is independent of $\mathbf{u}$.

In the linearized problem (Sect. 2) stability is present in the indicated wider meaning when $k \neq 0$ (see [5]). For $k = 0$ and $m \neq 0$ there is no stability (see (2.4)) and, consequently, it does not exist (in three-dimensional analysis) in the exact nonlinear meaning.

For other stationary flows of perfect fluid stability in the wider meaning is not excluded. It is, however, possible to expect that in the conventional meaning (Sect. 1) all stationary flows of perfect fluid in a channel, tube, or between two rotating cylinders are (at least weakly) unstable.

Notes. 1) (to Sect. 1). The Liapunov stability was apparently established only in the particular case of perfect fluid flows in a channel, in which convex profiles of $U_0(y)$, only plane perturbations and norm $\| \mathbf{u} \|_3$ are considered. In that case stability exists not only in the meaning of (1.2) but, also, in its exact (nonlinear) Liapunov's definition [6]. (In linear analysis the first condition may be somewhat relaxed).

2) (to Sect. 2). The three-dimensional instability evident in the considered problem in the meaning of $\| \mathbf{u} \|_3$ (or $\| \operatorname{rot} \mathbf{u} \|_2$) is probably a particular manifestation of the general tendency of three-dimensional (rotational) to vorticity increase. For stationary flows of the "general kind" this tendency is discussed (but not proved) in [7].

3) (to Sect. 3). The absolute continuity of spectrum A is a comparatively refined result. Here a less subtle one is sufficient. Let $A_0\gamma = -kU_0(y)\gamma$ and $A = A_0 + B$. The following reasoning (usual in the theory of perturbations in a continuous spectrum) shows that for certain γ_0 we have the asymptotic approach $e^{itA}\gamma_0 \sim e^{itA_0}h$. Let $\psi = e^{-itA}e^{itA_0}h$. Then

$$\frac{d\psi}{dt} = -ie^{-itA}Be^{itA_0}h, \qquad \left\|\frac{d\psi}{dt}\right\| \leqslant C\|Be^{itA_0}h\|$$

Let h be twice continuously differentiable and vanish in some neighborhood of points $0, 1, y_c$ (where $U_0'(y_c) = 0$). Then

$$\|Be^{itA_0}h\| \leqslant C/t^2, \qquad \psi(t) = \psi_\infty + O(1/t)$$

Let us set $\gamma_0 = \psi_\infty$. Then

$$\gamma(t) = e^{itA}\gamma_0 = e^{itA_0}h + \varepsilon = e^{-iktU_0(y)}h(y) + \varepsilon(y, t), \; \|\varepsilon\| \leqslant C/t$$

$$v'(y, t) = \int_0^1 \frac{\partial G}{\partial y}(y, \eta)\gamma(\eta, t)\,d\eta = J + \varepsilon_1$$

$$J = \int_0^1 \frac{\partial G}{\partial y}(y, \eta)\,e^{-iktU_0(\eta)}h(\eta)\,d\eta$$

where $G(y, \eta)$ is the Green's function for

$$v'' - (k^2 + m^2)v = f, \quad v(0) = v(1) = 0$$

Since the integral J tends uniformly to zero $(= O(1/t))$, with respect to y, hence

$$\|v\|_2 = \|v'\|_2 \leqslant C/t$$

4) (to Sect. 4). In the often cited works [2, 3, 8] plane perturbations $(m = 0)$ are considered and the boundedness of individual solutions of linearized equations for $t \to \infty$ is discussed with the use of finiteness of $|\operatorname{grad} \mathbf{u}|_{t=0}$. It should be stresses that, if the boundedness for $t \to \infty$ is specified for all initial data, Liapunov's stability follows (in the linear approximation) in the meaning of Sect. 4 but not in the conventional meaning (Sect. 1) (*).

5) (to Sects. 1–3). Equations (2.1) have solutions for any initial data in L_2, which satisfy the condition $iku + v' + imw = 0$. Hence there exist individual solutions of $\mathbf{u}(y, t)$ for which $\|\mathbf{u}(y, t)\| \to \infty$. Evidently such solutions cannot be smooth, since for these $v'' \notin L_2$. By the same token $\mathbf{u}$ has an "infinite vorticity": one of the integrals

$$\int_0^1 |u'|^2 dy, \qquad \int_0^1 |w'|^2 dy$$

does not exist. However there exist simultaneously $\mathbf{u}_n(y, t)$ which are as smooth as required and satisfy (1.1).

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*) Shnol' E.E. On the theory of stability of the simplest stationary flows of perfect fluid. Preprint (in Russian), Inst. Prikl. Matem., Akad. Nauk SSSR, № 53, 1973.

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SIMULATION OF CASCADE PROCESSES IN TURBULENT FLOWS

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Model equations are derived for collective degrees of freedom, i. e. Fourier amplitudes of velocity field summated over the wave number octave (the wave number modulus changes twice within the octave). Stationary solutions of these equations which in the related inertial intervals yield the laws of similarity are analyzed ($k^{-5/3}$ in a three-dimensional turbulence and k^{-3} in a two-dimensional one). Nonstationary problems of forming cascade processes were numerically investigated in [1].

Simulation of cascade processes of energy transmission, vorticity, nonuniform concentration of admixtures is of particular interest in investigations of turbulent flows by the spectrum of turbulent motions. Cascade processes determine the inner structure of flows and the mechanism of turbulent dissipation. In the last few years it has been possible to simulate on a computer a two-dimensional space-periodic flow of not very high viscosity and to obtain a section of the energy spectrum $E(k) \sim k^{-3}$ [2-5] which corresponds to the cascade process of vorticity transfer [2, 6]. The authors are aware of only one publication [7] on numerical simulation of three-dimensional periodic flows, where the Reynolds numbers were not sufficiently high for the investigation of the cascade energy transmission process and obtaining a section of the spectrum governed by the "law of $^5/_3$".